SKEW PIERI RULES FOR HALL–LITTLEWOOD FUNCTIONS

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Abstract. We produce skew Pieri Rules for Hall–Littlewood functions in the spirit of Assaf and McNamara [3]. The first two were conjectured by the first author [6]. The key ingredients in the proofs are a q-binomial identity for skew partitions and a Hopf algebraic identity that expands products of skew elements in terms of the coproduct and the antipode.

Let $\Lambda[t]$ denote the ring of symmetric functions over $\mathbb{Q}(t)$, and let $\{s_\lambda\}$ and $\{P_\lambda(t)\}$ denote its bases of Schur functions and Hall–Littlewood functions, respectively, indexed by partitions λ. The Schur functions (which are actually defined over $\mathbb{Z}$) lead a rich life—making appearances in combinatorics, representation theory, and Schubert calculus, among other places. See [5, 9] for details. The Hall–Littlewood functions are nearly as ubiquitous (having as a salient feature that $P_\lambda(t) \to s_\lambda$ under the specialization $t \to 0$). See [8] and the references therein for their place in the literature.

A classical problem is to determine cancellation-free formulas for multiplication in these bases,

$$s_\lambda\, s_\mu \;=\; \sum_\nu c^{\,\nu}_{\lambda,\mu}\, s_\nu \quad \text{and} \quad P_\lambda\, P_\mu \;=\; \sum_\nu f^{\,\nu}_{\lambda,\mu}(t)\, P_\nu.$$

The first problem was only given a complete solution in the latter half of the 20th century, while the second problem remains open. Special cases of the problem, known as *Pieri rules,* have been understood for quite a bit longer.

The Pieri rules for Schur functions [9, Ch. I, (5.16) and (5.17)] take the form

$$s_\lambda\, s_{1^r} = s_\lambda\, e_r = \sum_{\lambda^+} s_{\lambda^+}\,, \tag{1}$$

with the sum over partitions λ^+ for which λ^+/λ is a vertical strip of size r, and

$$s_\lambda\, s_r = \sum_{\lambda^+} s_{\lambda^+}\,, \tag{2}$$

with the sum over partitions λ^+ for which λ^+/λ is a horizontal strip of size r. (See Section 1 for the definitions of vertical- and horizontal strip.)

The Pieri rules for Hall–Littlewood functions [9, Ch. III, (3.2) and (5.7)] state that

$$P_\lambda\, P_{1^r} = P_\lambda\, e_r = \sum_{|\lambda^+/\lambda|=r} \mathrm{vs}_{\lambda^+/\lambda}(t) P_{\lambda^+} \tag{3}$$

2010 *Mathematics Subject Classification.* 05E05; 05E10; 16T05; 16T30; 33D52.

Key words and phrases. Pieri Rules, Hall–Littlewood functions.

Matjaž Konvalinka was partially supported by Research Programs P1-0294 and P1-0297 of the Slovenian Research Agency.

Aaron Lauve was supported in part by NSA grant #H98230-10-1-0362.

and

$$P_\lambda\, q_r = \sum_{|\lambda^+/\lambda|=r} \mathrm{hs}_{\lambda^+/\lambda}(t) P_{\lambda^+}\,, \tag{4}$$

with the sums again running over vertical strips and horizontal strips, respectively. Here q_r denotes $(1-t)P_r$ for $r>0$ with $q_0 = P_0 = 1$, and $\mathrm{vs}_{\lambda/\mu}(t)$, $\mathrm{hs}_{\lambda/\mu}(t)$ are certain polynomials in t. (See Section 1 for their definitions, as well as those of $\mathrm{sk}_{\lambda/\mu}(t)$ and $\mathrm{br}_{\lambda/\mu}(t)$ appearing below.)

In many respects (beyond the obvious similarity of (2) and (4)), the q_r play the same role for Hall–Littlewood functions that the s_r play for Schur functions. Still, one might ask for a link between the two theories. The following generalization of (2), which seems to be missing from the literature, is our first result (Section 1).

Theorem 1. *For a partition λ and $r \geq 0$, we have*

$$P_\lambda\, s_r = \sum_{\lambda^+} \mathrm{sk}_{\lambda^+/\lambda}(t) P_{\lambda^+}\,, \tag{5}$$

with the sum over partitions $\lambda^+ \supseteq \lambda$ for which $|\lambda^+/\lambda| = r$.

The main focus of this article is on the generalizations of Hall–Littlewood functions to skew shapes λ/μ. Our specific question about skew Hall–Littlewood functions is best introduced via the recent answer for skew Schur functions $s_{\lambda/\mu}$. In [3], Assaf and McNamara give a *skew Pieri rule* for Schur functions. They prove (bijectively) the following generalization of (2):

$$s_{\lambda/\mu}\, s_r = \sum_{\lambda^+,\,\mu^-} (-1)^{|\mu/\mu^-|} s_{\lambda^+/\mu^-}\,, \tag{6}$$

with the sum over pairs (λ^+, μ^-) of partitions such that λ^+/λ is a horizontal strip, μ/μ^- is a vertical strip, and $|\lambda^+/\lambda| + |\mu/\mu^-| = r$. This elegant gluing-together of an s_r-type Pieri rule for the outer rim of λ/μ with an e_r-type Pieri rule for the inner rim of λ/μ demanded further exploration.

Before we survey the literature that followed the Assaf–McNamara result, we call attention to some work that preceded it. The skew Schur functions do not form a basis; so, from a strictly ring theoretic perspective (or representation theoretic, or geometric), it is more natural to ask how the product in (6) expands in terms of Schur functions. This answer, and vast generalizations of it, was provided by Zelevinsky in [12]. In fact, (6) provides such an answer as well, since

$$s_{\lambda^+/\mu^-} = \sum_\nu c^{\lambda^+}_{\mu^-,\nu}\, s_\nu$$

and the coefficients $c^{\lambda^+}_{\mu^-,\nu}$ are well-understood, but the resulting formula has an enormous amount of cancellation, while Zelevinsky's is cancellation free. It is an open problem to find a representation theoretic (or geometric) explanation of (6).

Remark. As an example of the type of explanation we mean, recall Zelevinsky's realization [13] of the classical Jacobi–Trudi formula for s_λ ($\lambda \vdash n$) from the resolution of a well-chosen polynomial representation of GL_n. See also [1, 4].

Returning to the literature that followed [3], Lam, Sottile, and the second author [7] found a Hopf algebraic explanation for (6) that readily extended to many other settings. A skew Pieri rule for k-Schur functions was given, for instance, as well one for (noncommutative) ribbon Schur functions. Within the setting of Schur functions, it provided an easy extension

of (6) to products of arbitrary skew Schur functions—a formula first conjectured by Assaf and McNamara in [3]. (The results of this paper use the same Hopf machinery. For the non-experts, we reprise most of details and background in Section 2.)

Around the same time, the first author [6] was motivated to give a skew Murnaghan-Nakayama rule in the spirit of Assaf and McNamara. Along the way, he gives a bijective proof of the conjugate form of (6) (only proven in [3] using the automorphism ω) and a *quantum* skew Murnaghan-Nakayama rule that takes the following form.

$$s_{\lambda/\mu}\, q_r = \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|}\, \mathrm{br}_{\lambda^+/\lambda}(t)\, \mathrm{br}_{(\mu/\mu^-)^c}(t) s_{\lambda^+/\mu^-}\,, \tag{7}$$

with the sum over pairs (λ^+,μ^-) of partitions such that λ^+/λ and μ/μ^- are broken ribbons and $|\lambda^+/\lambda| + |\mu/\mu^-| = r$. Note that since $P_r(0) = s_r$, we recover the skew Pieri rule for $t = 0$. Also, since $P_r(1) = p_r$ (the r-th power sum symmetric function), we recover the skew Murnaghan-Nakayama rule [2] if we divide the formula by $1-t$ and let $t \to 1$. This formula, like that in Theorem 1, may be viewed as a link between the two theories of Schur and Hall–Littlewood functions. One is tempted to ask for other examples of mixing, e.g., swapping the rolls of Schur and Hall–Littlewood functions in (7). Two such examples were found (conjecturally) in [6]. Their proofs, and a generalization of (6) to the Hall–Littlewood setting, are the main results of this paper.

Theorem 2. *For partitions λ,μ, $\mu \subseteq \lambda$, and $r \geq 0$, we have*

$$P_{\lambda/\mu}\, s_{1^r} = P_{\lambda/\mu}\, e_r = P_{\lambda/\mu}\, P_{1^r} = \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|}\, \mathrm{vs}_{\lambda^+/\lambda}(t)\, \mathrm{sk}_{\mu/\mu^-}(t)\, P_{\lambda^+/\mu^-}\,,$$

where the sum on the right is over all $\lambda^+ \supseteq \lambda$, $\mu^- \subseteq \mu$ such that $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

Theorem 3. *For partitions λ,μ, $\mu \subseteq \lambda$, and $r \geq 0$, we have*

$$P_{\lambda/\mu}\, s_r = \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|}\, \mathrm{sk}_{\lambda^+/\lambda}(t)\, \mathrm{vs}_{\mu/\mu^-}(t)\, P_{\lambda^+/\mu^-}\,,$$

where the sum on the right is over all $\lambda^+ \supseteq \lambda$, $\mu^- \subseteq \mu$ such that $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

Note that putting $\mu = \emptyset$ above recovers Theorem 1. (We offer two proofs of Theorem 3; one that rests on Theorem 1 and one that does not.)

Theorem 4. *For partitions λ,μ, $\mu \subseteq \lambda$, and $r \geq 0$, we have*

$$P_{\lambda/\mu}\, q_r = \sum_{\lambda^+,\mu^-,\nu} (-1)^{|\mu/\mu^-|}(-t)^{|\nu/\mu^-|}\, \mathrm{hs}_{\lambda^+/\lambda}(t)\, \mathrm{vs}_{\mu/\nu}(t)\ \mathrm{sk}_{\nu/\mu^-}(t)\, P_{\lambda^+/\mu^-}\,,$$

where the sum on the right is over all $\lambda^+ \supseteq \lambda$, $\mu^- \subseteq \nu \subseteq \mu$ such that $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

Remark. We reiterate that the skew elements do not form a basis for $\Lambda[t]$, so the expansions announced in Theorems 2–4 are by no means unique. However, if we demand that the expansions be over partitions $\lambda^+ \supseteq \lambda$ and $\mu^- \subseteq \mu$, and that the coefficients factor nicely as products of polynomials $a_{\lambda^+/\lambda}(t)$ (independent of μ) and $b_{\mu/\mu^-}(t)$ (independent of λ), then they are in fact unique (up to scalar). We make this remark precise in Theorem 12 in Section 3.

This paper is organized as follows. In Section 1, we prove some polynomial identities involving hs, vs and sk, prove Theorem 1, and find $\omega(q_r)$. In Section 2, we introduce our main tool, Hopf algebras. We conclude in Section 3 with the proofs of our main theorems.

1. Combinatorial Preliminaries

1.1. **Notation, and a key lemma.** The conjugate partition of λ is denoted λ^c. We write $m_i(\lambda)$ for the number of parts of λ equal to i. The q-binomial coefficient is defined by

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{(1-q^a)(1-q^{a-1})\cdots(1-q^{a-b+1})}{(1-q^b)(1-q^{b-1})\cdots(1-q)}$$

and is a polynomial in q that gives $\binom{a}{b}$ when $q=1$. For a partition λ, define $n(\lambda) = \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda_i^c}{2}$.

Given two partitions λ and μ, we say $\mu \subseteq \lambda$ if $\lambda_i \geq \mu_i$ for all $i \geq 1$, in which case we may consider the pair as a *skew shape* λ/μ. We write $[\lambda/\mu]$ for the cells $\{(i,j)\colon 1 \leq i \leq \ell(\lambda),\ \mu_i < j \leq \lambda_i\}$. We say that λ/μ is a *horizontal strip* (respectively *vertical strip*) if $[\lambda/\mu]$ contains no 2×1 (respectively 1×2) block, equivalently, if $\lambda_i^c \leq \mu_i^c + 1$ (respectively $\lambda_i \leq \mu_i + 1$) for all i. We say that λ/μ is a *ribbon* if $[\lambda/\mu]$ is connected and if it contains no 2×2 block, and that λ/μ is a *broken ribbon* if $[\lambda/\mu]$ contains no 2×2 block, equivalently, if $\lambda_i \leq \mu_{i-1}+1$ for $i \geq 2$. The Young diagram of a broken ribbon is a disjoint union of $\operatorname{rib}(\lambda/\mu)$ number of ribbons. The *height* $\operatorname{ht}(\lambda/\mu)$ (respectively *width* $\operatorname{wt}(\lambda/\mu)$) of a ribbon is the number of non-empty rows (respectively columns) of $[\lambda/\mu]$, minus 1. The height (respectively width) of a broken ribbon is the sum of heights (respectively widths) of the components.

Let us define some polynomials. For a horizontal strip λ/μ, define

$$\operatorname{hs}_{\lambda/\mu}(t) = \prod_{\substack{\lambda_j^c = \mu_j^c + 1 \\ \lambda_{j+1}^c = \mu_{j+1}^c}} (1 - t^{m_j(\lambda)}).$$

If λ/μ is not a horizontal strip, define $\operatorname{hs}_{\lambda/\mu}(t) = 0$. For a vertical strip λ/μ, define

$$\operatorname{vs}_{\lambda/\mu}(t) = \prod_{j \geq 1} \begin{bmatrix} \lambda_j^c - \lambda_{j+1}^c \\ \lambda_j^c - \mu_j^c \end{bmatrix}_t.$$

If λ/μ is not a vertical strip, define $\operatorname{vs}_{\lambda/\mu}(t) = 0$. For a broken ribbon λ/μ, define

$$\operatorname{br}_{\lambda/\mu}(t) = (-t)^{\operatorname{ht}(\lambda/\mu)}(1-t)^{\operatorname{rib}(\lambda/\mu)}.$$

If λ/μ is not a broken ribbon, define $\operatorname{br}_{\lambda/\mu}(t) = 0$. For any skew shape λ/μ, define

$$\operatorname{sk}_{\lambda/\mu}(t) = t^{\sum_j \binom{\lambda_j^c - \mu_j^c}{2}} \prod_{j \geq 1} \begin{bmatrix} \lambda_j^c - \mu_{j+1}^c \\ m_j(\mu) \end{bmatrix}_t.$$

Next, recall the *q-binomial theorem.* For all $n, k \geq 0$, we have

$$\prod_{i=0}^{n-1}(t+q^i) = \sum_{k=0}^{n} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k. \tag{8}$$

This may be proven by induction from the standard identity $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$.

Lemma 5. *For fixed partitions λ, μ satisfying $\mu \subseteq \lambda$, we have*

$$\sum_{\nu} (-t)^{|\lambda/\nu|} \operatorname{vs}_{\lambda/\nu}(t) \operatorname{sk}_{\nu/\mu}(t) = \operatorname{hs}_{\lambda/\mu}(t),$$

with the sum over all ν, $\mu \subseteq \nu \subseteq \lambda$, for which λ/ν is a vertical strip.

Proof. Let $a_j = \lambda_j^c - \max(\mu_j^c, \lambda_{j+1}^c) \geq 0$. A partition ν, $\mu \subseteq \nu \subseteq \lambda$, for which λ/ν is a vertical strip is obtained by choosing k_j, $0 \leq k_j \leq a_j$, and removing k_j bottom cells of column j in λ. See Figure 1 for the example for $\lambda = 98886666444$ and $\mu = 77666633331$, where $a_4 = 3$, $a_6 = 2$, $a_8 = 3$, $a_9 = 1$ and $a_i = 0$ for all other i.

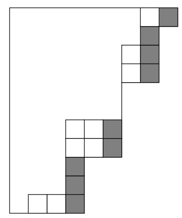


FIGURE 1. A partition ν ($\mu \subseteq \nu \subseteq \lambda$) for which λ/ν is a vertical strip within λ/μ is built from λ by removing some number of the shaded cells of $[\lambda]$.

We have $|\lambda/\nu| = \sum_j k_j$, $\nu_j^c = \lambda_j^c - k_j$. The choices of the k_j are independent, which means that

$$\sum_\nu (-t)^{|\lambda/\nu|}\, \mathrm{sk}_{\nu/\mu}(t)\, \mathrm{vs}_{\lambda/\nu}(t) = \sum_{k_1,k_2,\dots} (-t)^{\sum_j k_j} t^{\sum_j \binom{\nu_j^c-\mu_j^c}{2}} \prod_j \begin{bmatrix} \nu_j^c - \mu_{j+1}^c \\ m_j(\mu) \end{bmatrix}_t \prod_j \begin{bmatrix} \lambda_j^c - \lambda_{j+1}^c \\ \lambda_j^c - \nu_j^c \end{bmatrix}_t$$

$$= \prod_j \sum_{k_j=0}^{a_j} (-t)^{k_j} t^{\binom{\lambda_j^c-\mu_j^c-k_j}{2}} \begin{bmatrix} \lambda_j^c - k_j - \mu_{j+1}^c \\ m_j(\mu) \end{bmatrix}_t \begin{bmatrix} m_j(\lambda) \\ k_j \end{bmatrix}_t. \tag{9}$$

We analyze (9) case-by-case, showing that it reduces to $\mathrm{hs}_{\lambda/\mu}(t)$ when λ/μ is a horizontal strip and zero otherwise. Assume first that λ/μ is a horizontal strip. This means that $a_j \leq \lambda_j^c - \mu_j^c \leq 1$ for all j.

Case 1: $a_j = 0$. We have $\max(\mu_j^c, \lambda_{j+1}^c) = \lambda_j^c$, so the inner sum in (9) is equal to

$$\begin{bmatrix} \lambda_j^c - \mu_{j+1}^c \\ m_j(\mu) \end{bmatrix}_t = \begin{bmatrix} \lambda_j^c - \mu_{j+1}^c \\ \mu_j^c - \mu_{j+1}^c \end{bmatrix}_t.$$

If $\mu_j^c = \lambda_j^c$, this is 1, and if $\mu_j^c = \lambda_j^c - 1$ and $\lambda_{j+1}^c = \lambda_j^c$, then $\mu_{j+1}^c = \mu_j^c$ and so the expression is also 1.

Case 2: $a_j = 1$. This holds if and only if $\lambda_j^c = \mu_j^c + 1$, $\lambda_{j+1}^c \leq \lambda_j^c - 1$, in which case the sum in (9) is

$$(-t)^0 t^{\binom{1}{2}} \begin{bmatrix} 1 + m_j(\mu) \\ m_j(\mu) \end{bmatrix}_t \begin{bmatrix} m_j(\lambda) \\ 0 \end{bmatrix}_t + (-t)^1 t^{\binom{0}{2}} \begin{bmatrix} m_j(\mu) \\ m_j(\mu) \end{bmatrix}_t \begin{bmatrix} m_j(\lambda) \\ 1 \end{bmatrix}_t$$

$$= 1 + t + \ldots + t^{m_j(\mu)} - t\big(1 + t + \ldots + t^{m_j(\lambda)-1}\big) = \begin{cases} 1 - t^{m_j(\lambda)} & : \ \lambda_j^c = \mu_j^c + 1, \lambda_{j+1}^c = \mu_{j+1}^c \\ 1 & : \ \text{otherwise} \end{cases}.$$

Indeed, $\lambda_j^c = \mu_j^c + 1$ and $\lambda_{j+1}^c = \mu_{j+1}^c + 1$ imply $m_j(\mu) = m_j(\lambda)$, while $\lambda_j^c = \mu_j^c + 1$ and $\lambda_{j+1}^c = \mu_{j+1}^c$ imply $\lambda_{j+1}^c \leq \mu_j^c = \lambda_j^c - 1$ and $m_j(\mu) = m_j(\lambda) - 1$. Thus (9) equals $\mathrm{hs}_{\lambda/\mu}(t)$ whenever λ/μ is a horizontal strip.

Now assume that λ/μ is not a horizontal strip. Let j be the largest index for which $\lambda_j^c - \mu_j^c \geq 2$. Let us investigate two cases, when $\lambda_{j+1}^c > \mu_j^c$ and when $\lambda_{j+1}^c \leq \mu_j^c$.

Case 1: $\lambda_{j+1}^c > \mu_j^c$. We must have $\lambda_{j+1}^c = \mu_j^c + 1$ and $\mu_{j+1}^c = \mu_j^c$, for otherwise $\lambda_{j+1}^c - \mu_{j+1}^c = (\lambda_{j+1}^c - \mu_j^c) + (\mu_j^c - \mu_{j+1}^c) \geq 2$, which contradicts the maximality of j. So $a_j = m_j(\lambda)$, $\lambda_j^c - \mu_j^c = \lambda_j^c - \mu_{j+1}^c = m_j(\lambda) + 1$, $m_j(\mu) = 0$, $m_j(\lambda) \geq 1$ and

$$\sum_{k_j=0}^{a_j} (-t)^{k_j} t^{\binom{\lambda_j^c - \mu_j^c - k_j}{2}} \begin{bmatrix} \lambda_j^c - k_j - \mu_{j+1}^c \\ m_j(\mu) \end{bmatrix}_t \begin{bmatrix} m_j(\lambda) \\ k_j \end{bmatrix}_t = \sum_{k_j=0}^{m_j(\lambda)} (-t)^{k_j} t^{\binom{m_j(\lambda)+1-k_j}{2}} \begin{bmatrix} m_j(\lambda) \\ k_j \end{bmatrix}_t$$
$$= \sum_{k_j=0}^{m_j(\lambda)} (-t)^{k_j} t^{\binom{m_j(\lambda)-k_j}{2} + m_j(\lambda) - k_j} \begin{bmatrix} m_j(\lambda) \\ k_j \end{bmatrix}_t = t^{m_j(\lambda)} \sum_{k_j=0}^{m_j(\lambda)} (-1)^{k_j} t^{\binom{m_j(\lambda)-k_j}{2}} \begin{bmatrix} m_j(\lambda) \\ k_j \end{bmatrix}_t.$$

Using (8) with $n = m_j(\lambda)$, $t = -1$ and $q = t$, the above simplifies to

$$t^{m_j(\lambda)} \prod_{i=0}^{m_j(\lambda)-1} (-1 + t^i) = 0.$$

Case 2: $\lambda_{j+1}^c \leq \mu_j^c$. We consider two further options. If $\mu_{j+1}^c = \lambda_{j+1}^c$, then $a_j = \lambda_j^c - \mu_j^c = m_j(\lambda) - m_j(\mu) \geq 2$ and

$$\sum_{k_j=0}^{a_j} (-t)^{k_j} t^{\binom{\lambda_j^c - \mu_j^c - k_j}{2}} \begin{bmatrix} \lambda_j^c - k_j - \mu_{j+1}^c \\ m_j(\mu) \end{bmatrix}_t \begin{bmatrix} m_j(\lambda) \\ k_j \end{bmatrix}_t$$
$$= \sum_{k_j=0}^{m_j(\lambda)-m_j(\mu)} (-t)^{k_j} t^{\binom{m_j(\lambda)-m_j(\mu)-k_j}{2}} \begin{bmatrix} m_j(\lambda) - k_j \\ m_j(\mu) \end{bmatrix}_t \begin{bmatrix} m_j(\lambda) \\ k_j \end{bmatrix}_t$$
$$= \sum_{k_j=0}^{m_j(\lambda)-m_j(\mu)} (-t)^{k_j} t^{\binom{m_j(\lambda)-m_j(\mu)-k_j}{2}} \begin{bmatrix} m_j(\lambda) - m_j(\mu) \\ k_j \end{bmatrix}_t \begin{bmatrix} m_j(\lambda) \\ m_j(\mu) \end{bmatrix}_t.$$

If we use (8) with $n = m_j(\lambda) - m_j(\mu)$, $t = -t$ and $q = t$, we get

$$\begin{bmatrix} m_j(\lambda) \\ m_j(\mu) \end{bmatrix}_t \prod_{i=0}^{m_j(\lambda)-m_j(\mu)-1} (-t + t^i) = 0.$$

On the other hand, if $\mu_{j+1}^c = \lambda_{j+1}^c - 1$, then $a_j = \lambda_j^c - \mu_j^c = m_j(\lambda) - m_j(\mu) + 1 \geq 2$ and

$$\sum_{k_j=0}^{a_j} (-t)^{k_j} t^{\binom{\lambda_j^c - \mu_j^c - k_j}{2}} \begin{bmatrix} \lambda_j^c - k_j - \mu_{j+1}^c \\ m_j(\mu) \end{bmatrix}_t \begin{bmatrix} m_j(\lambda) \\ k_j \end{bmatrix}_t$$
$$= \sum_{k_j=0}^{m_j(\lambda)-m_j(\mu)+1} (-t)^{k_j} t^{\binom{m_j(\lambda)-m_j(\mu)+1-k_j}{2}} \begin{bmatrix} m_j(\lambda) + 1 - k_j \\ m_j(\mu) \end{bmatrix}_t \begin{bmatrix} m_j(\lambda) \\ k_j \end{bmatrix}_t$$
$$= \sum_{k_j=0}^{m_j(\lambda)-m_j(\mu)+1} (-t)^{k_j} t^{\binom{m_j(\lambda)-m_j(\mu)+1-k_j}{2}} \frac{1 - t^{m_j(\lambda)+1-k_j}}{1 - t^{m_j(\lambda)-m_j(\mu)+1}} \begin{bmatrix} m_j(\lambda) - m_j(\mu) + 1 \\ k_j \end{bmatrix}_t \begin{bmatrix} m_j(\lambda) \\ m_j(\mu) \end{bmatrix}_t$$

$$= \frac{1}{1-t^{m_j(\lambda)-m_j(\mu)+1}} \begin{bmatrix} m_j(\lambda) \\ m_j(\mu) \end{bmatrix}_t \left(\sum_{k_j=0}^{m_j(\lambda)-m_j(\mu)+1} (-t)^{k_j} t^{\binom{m_j(\lambda)-m_j(\mu)+1-k_j}{2}} \begin{bmatrix} m_j(\lambda)-m_j(\mu)+1 \\ k_j \end{bmatrix}_t \right.$$
$$\left. - \sum_{k_j=0}^{m_j(\lambda)-m_j(\mu)+1} (-1)^{k_j} t^{\binom{m_j(\lambda)-m_j(\mu)+1-k_j}{2}} t^{m_j(\lambda)+1} \begin{bmatrix} m_j(\lambda)-m_j(\mu)+1 \\ k_j \end{bmatrix}_t \right).$$

We prove that the first (respectively, second) sum is 0 by substituting $n = m_j(\lambda)-m_j(\mu)+1$, $t=-t$ (respectively, $t=-1$) and $q=t$ in (8). This finishes the proof of the lemma. ☐

1.2. **Elementary Hall–Littlewood identities.** We give two applications of Lemma 5, then prove some elementary properties on Hall–Littlewood functions that will be useful in Section 3. The first application is a formula for the product of a Hall–Littlewood polynomial with the Schur function s_r.

Proof of Theorem 1. The proof is by induction on r. For $r=0$, there is nothing to prove. For $r>0$, we use the formula

$$q_r = \sum_{k=0}^{r} (-t)^k s_{r-k} e_k, \tag{10}$$

which is proven as follows. It is well-known and easy to prove (see e.g. [11, Exercise 7.11]) that

$$P_r = \sum_{\tau \vdash n} (1-t)^{\ell(\tau)-1} m_\tau = \sum_{k=0}^{r-1} (-t)^k s_{r-k,1^k}\,.$$

The conjugate Pieri rule then gives (10), for

$$\sum_{k=0}^{r} (-t)^k s_{r-k} e_k = s_r + \sum_{k=1}^{r-1} (-t)^k (s_{r-k,1^k} + s_{r-k+1,1^{k-1}}) + (-t)^r s_{1^r} = q_r\,.$$

For $|\lambda^+/\lambda| = r$, the coefficient of P_{λ^+} in

$$P_\lambda\, s_r = P_\lambda \left(q_r - \sum_{k=1}^{r} (-t)^k s_{r-k} e_k \right)$$

reduces by induction, (3) and (4) to

$$\mathrm{hs}_{\lambda^+/\lambda}(t) - \sum (-t)^{|\lambda^+/\nu|}\, \mathrm{sk}_{\nu/\lambda}(t)\, \mathrm{vs}_{\lambda^+/\nu}(t),$$

with the sum over all ν, $\lambda \subseteq \nu \subseteq \lambda^+$, for which λ^+/ν is a vertical strip of size at least 1. By Lemma 5, this is equal to $\mathrm{sk}_{\lambda^+/\lambda}(t)$. ☐

Recall that $f^\lambda_{\mu,\tau}(t)$ is the (polynomial) coefficient of P_λ in $P_\mu P_\tau$.

Corollary 6. *The structure constants $f^\lambda_{\mu,\tau}(t)$ satisfy* $\displaystyle\sum_\tau t^{n(\tau)} f^\lambda_{\mu,\tau}(t) = \mathrm{sk}_{\lambda/\mu}(t)$.

Proof. This follows from $s_r = \sum_{\tau \vdash r} t^{n(\tau)} P_\tau$, which is (2) in [9, page 219] and also Theorem 1 for $\lambda = \emptyset$. ☐

The second application of Lemma 5 is the following generalization of Example 1 of [9, §III.3, Example 1].

Theorem 7. *For every λ, μ, we have*

$$\sum_{\nu} \mathrm{vs}_{\lambda/\nu}(t)\,\mathrm{sk}_{\nu/\mu}(t) y^{|\lambda/\nu|} = \sum_{\sigma} t^{n(\sigma)-\binom{\ell(\sigma)}{2}} f^{\lambda}_{\sigma\mu}(t) \prod_{j=1}^{\ell(\sigma)} (y + t^{j-1}). \tag{11}$$

Equivalently, for all m,

$$\sum_{\nu\colon |\lambda/\nu|=m} \mathrm{vs}_{\lambda/\nu}(t)\,\mathrm{sk}_{\nu/\mu}(t) = \sum_{\sigma} t^{n(\sigma)-\binom{m}{2}} f^{\lambda}_{\sigma\mu}(t) \begin{bmatrix}\ell(\sigma)\\ m\end{bmatrix}_{t^{-1}}. \tag{12}$$

Proof. Let us evaluate $P_\mu\, s_r \left(\sum_m e_m\, y^m\right)$ in two different ways. On the one hand,

$$P_\mu\, s_r \left(\sum_m e_m\, y^m\right) = \left(\sum_\nu \mathrm{sk}_{\nu/\mu}(t) P_\nu\right)\left(\sum_m e_m\, y^m\right) = \sum_{\nu,\lambda} \mathrm{sk}_{\nu/\mu}(t)\,\mathrm{vs}_{\lambda/\nu}(t) P_\lambda\, y^{|\lambda/\nu|}.$$

On the other hand, using Example 1 on page 218 of [9],

$$P_\mu\, s_r \left(\sum_m e_m y^m\right) = P_\mu \sum_\sigma t^{n(\sigma)} P_\sigma \prod_{j=1}^{\ell(\sigma)} (1 + t^{1-j} y) = \sum_{\sigma,\lambda} t^{n(\sigma)-\binom{\ell(\sigma)}{2}} f^{\lambda}_{\sigma\mu}(t) P_\lambda \prod_{j=1}^{\ell(\sigma)} (y + t^{j-1}).$$

Now (11) follows by taking the coefficient of P_λ in both expressions. For (12), we use the q-binomial theorem (8) and

$$\begin{bmatrix}n\\ k\end{bmatrix}_{t^{-1}} = t^{\binom{k}{2}+\binom{n-k}{2}-\binom{n}{2}} \begin{bmatrix}n\\ k\end{bmatrix}_{t}.$$

□

Remark. The theorem is indeed a generalization of [9, §III.3, Example 1]. For $\mu = \emptyset$, $\mathrm{sk}_{\nu/\mu}(t) = t^{n(\nu)}$, and the right-hand side of (12) is non-zero only for $\sigma = \lambda$, so the last equation on page 218 (*loc. cit.*) follows. It also generalizes Lemma 5: for $y = -t$, the right-hand side of (11) is non-zero if and only if $\ell(\sigma) = 1$, and is therefore equal to $\mathrm{hs}_{\lambda/\mu}(t)$.

We finish the section with two more lemmas.

Lemma 8. *Given $r > k \geq 0$, we have*

$$s_{r-k,1^k} = \sum_{\lambda\colon \ell(\lambda)\geq k+1} t^{\binom{\ell(\lambda)-k}{2}+\sum_{i=2}^{\lambda_1}\binom{\lambda_i^c}{2}} \begin{bmatrix}\ell(\lambda)-1\\ k\end{bmatrix}_t P_\lambda.$$

Proof. The lemma follows from a formula due to Lascoux and Schützenberger. See [9, Ch. III, (6.5)]. In that terminology, we have to evaluate $K_{(r-k,1^k),\lambda}(t)$. We choose a semistandard Young tableau T of shape $(r-k, 1^k)$ and type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. Clearly, such tableaux are in one-to-one correspondence with k-subsets of the set $\{2, \ldots, \ell\}$. For such a subset S, write s for the word with the elements of S in increasing order, and write $\overline{s}$ for the word with the elements of $\{2, \ldots, \ell\} \setminus S$ in decreasing order. The reverse reading word of the tableau corresponding to S is $\ell^{\lambda_\ell - 1} \cdots 3^{\lambda_3-1} 2^{\lambda_2-1} 1^{\lambda_1} s$. The subwords $w_2, w_3, \ldots$ are all strictly decreasing, and $w_1 = \overline{s} 1 s$. The charges of $w_2, w_3, \ldots$ are $\binom{\lambda_2^c}{2}, \binom{\lambda_3^c}{2}, \ldots$, while the charge of w_1 is $\sum_{i\notin S}(\ell - i + 1)$ (sum over $i \notin S$, $2 \leq i \leq \ell$). We have

$$\sum_{S\subseteq\{2,\ldots,\ell+1\},|S|=k} t^{\sum_{i\notin S}(\ell+1-i+1)} = \sum_{S\subseteq\{2,\ldots,\ell\},|S|=k-1} t^{\sum_{i\notin S}(\ell+1-i+1)} + \sum_{S\subseteq\{2,\ldots,\ell\},|S|=k} t^{1+\sum_{i\notin S}(\ell+1-i+1)},$$

and the formula

$$\sum_{S\subseteq\{2,\dots,\ell\},|S|=k} t^{\sum_{i\notin S}(\ell-i+1)} = t^{\binom{\ell-k}{2}}\begin{bmatrix}\ell-1\\k\end{bmatrix}_t$$

follows by induction on ℓ. This finishes the proof. □

Lemma 9. *Let ω be the fundamental involution on $\Lambda[t]$ defined by $\omega(s_\lambda)=s_{\lambda^c}$. We have*

$$\omega(q_r)=(-1)^r\sum_{\lambda\vdash r}c_\lambda(t)P_\lambda,$$

where

$$c_\lambda(t)=t^{\sum_{i=2}^{\lambda_1}\binom{\lambda_i^c+1}{2}}\prod_{i=1}^{\ell(\lambda)}(-1+t^i).$$

Proof. We have

$$\begin{aligned}\omega(P_r)&=\omega\left(\sum_{k=0}^{r-1}(-t)^{r-k-1}s_{k+1,1^{r-k-1}}\right)=\sum_{k=0}^{r-1}(-t)^{r-k-1}s_{r-k,1^k}=\\&=\sum_{k=0}^{r-1}(-t)^{r-k-1}\left(\sum_{\ell(\lambda)\geq k+1}t^{\binom{\ell(\lambda)-k}{2}+\sum_{i=2}^{\lambda_1}\binom{\lambda_i^c}{2}}\begin{bmatrix}\ell(\lambda)-1\\k\end{bmatrix}_t P_\lambda\right)=\\&=\sum_{\lambda\vdash r}\left(\sum_{k=0}^{\ell(\lambda)-1}(-t)^{r-k-1}t^{\binom{\ell(\lambda)-k}{2}+\sum_{i=2}^{\lambda_1}\binom{\lambda_i^c}{2}}\begin{bmatrix}\ell(\lambda)-1\\k\end{bmatrix}_t\right)P_\lambda.\end{aligned}$$

Now by the q-binomial theorem,

$$\prod_{i=2}^{\ell(\lambda)}(-1+t^i)=t^{2(\ell(\lambda)-1)}\prod_{i=0}^{\ell(\lambda)-2}(-1/t^2+t^i)=t^{2(\ell(\lambda)-1)}\sum_{k=0}^{\ell(\lambda)-1}t^{\binom{\ell(\lambda)-1-k}{2}}\begin{bmatrix}\ell(\lambda)-1\\k\end{bmatrix}_t\left(-\frac{1}{t^2}\right)^k.$$

Simple calculations now show that the coefficient of P_λ in $\omega(q_r)=(1-t)\omega(P_r)$ is indeed $(-1)^r c_\lambda(t)$. □

2. Hopf Perspective on Skew Elements

Recall that $\Lambda[t]$ has another important basis $\{Q_\lambda\}$, defined by $Q_\lambda=b_\lambda(t)P_\lambda$, where $b_\lambda(t)=\prod_{i\geq1}(1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})$. The (extended) Hall scalar product on $\Lambda[t]$ is uniquely defined by either of the (equivalent) conditions

$$\langle P_\lambda,Q_\mu\rangle=\delta_{\lambda\mu}\quad\text{or}\quad\langle p_\lambda,p_\mu\rangle=z_\mu(t)\,\delta_{\lambda\mu}\,,$$

where, taking $\mu=(\mu_1,\mu_2,\dots,\mu_r)=\langle 1^{a_1},2^{a_2},\cdots,k^{a_k}\rangle$,

$$z_\mu(t)\;=\;z_\mu\cdot\prod_{j=1}^{r}(1-t^{\mu_j})^{-1}\;=\;\prod_{i=1}^{k}(i^{a_i}a_i!)\prod_{j=1}^{r}(1-t^{\mu_j})^{-1}\,.$$

See [9, §III.4]. The skew Hall–Littlewood function $P_{\lambda/\mu}$ is defined [9, Ch. III, (5.1′)] as the unique function satisfying

$$\left\langle P_{\lambda/\mu},Q_\nu\right\rangle=\langle P_\lambda,Q_\nu\,Q_\mu\rangle \tag{13}$$

for all $Q_\nu\in\Lambda[t]$. (Likewise for $Q_{\lambda/\mu}$.) If we choose to read $P_{\lambda/\mu}$ as, "Q_μ skews P_λ," then we allow ourselves access to the machinery of Hopf algebra actions on their duals. We introduce the basics in Subsection 2.1 and return to $\Lambda[t]$ and Hall–Littlewood functions in Subsection 2.2.

2.1. **Hopf preliminaries.** Let $H = \bigoplus_n H_n$ be a graded algebra over a field $\Bbbk$. Recall that H is a Hopf algebra if there are algebra maps $\Delta\colon H \to H \otimes H$, $\varepsilon\colon H \to \Bbbk$, and an algebra antimorphism $S\colon H \to H$, called the *coproduct*, *counit*, and *antipode*, respectively, satisfying some additional compatibility conditions. See [10].

Let $H^* = \bigoplus_n H_n^*$ denote the graded dual of H. If each H_n is finite dimensional, then the pairing $\langle\,\cdot\,,\,\cdot\,\rangle : H \otimes H^* \to \Bbbk$ defined by $\langle h, a\rangle = a(h)$ is nondegenerate. This pairing naturally endows H^* with a Hopf algebra structure, with product and coproduct uniquely determined by the formulas:

$$\langle h, a \cdot b\rangle := \langle \Delta(h), a \otimes b\rangle \quad \text{and} \quad \langle g \otimes h, \Delta(a)\rangle := \langle g \cdot h, a\rangle$$

for all homogeneous $g, h \in H$ and $a, b \in H^*$. (Extend to all of H^* by linearity, insisting that $\langle H_n, H_m^*\rangle = 0$ for $n \neq m$.)

Remark. The finite dimensionality of H_n ensures that the coproduct in H^* is a finite sum of functionals, $\Delta(a) = \sum_{(a)} a' \otimes a''$. Here and below we use Sweedler's notation for coproducts.

We now recall some standard actions ("$\rightharpoonup$") of H and H^* on each other. Given $h \in H$ and $a \in H^*$, put

$$a \rightharpoonup h := \sum_{(h)} \big\langle h'', a\big\rangle\, h' \quad \text{and} \quad h \rightharpoonup a := \sum_{(a)} \big\langle h, a''\big\rangle\, a'. \tag{14}$$

Equivalently, $\langle g, h \rightharpoonup a\rangle = \langle g \cdot h, a\rangle$ and $\langle a \rightharpoonup h, b\rangle = \langle h, b \cdot a\rangle$. We call these *skew elements* (in H and H^*, respectively) to keep the nomenclature consistent with that in symmetric function theory.

Our skew Pieri rules (Theorems 2, 3 and 4) come from an elementary formula relating products of elements h and skew elements $a \rightharpoonup g$ in a Hopf algebra H:

$$(a \rightharpoonup g) \cdot h = \sum \big(S(h'') \rightharpoonup a\big) \rightharpoonup (g \cdot h'). \tag{15}$$

See $(*)$ in the proof of [10, Lemma 2.1.4] or [7, Lemma 1]. Before turning to the proofs of these theorems, we first recall the Hopf structure of $\Lambda[t]$.

2.2. **The Hall–Littlewood setting.** The ring $\Lambda[t]$ is generated by the one-part power sum symmetric functions p_r $(r > 0)$, so the definitions

$$\Delta(p_r) := 1 \otimes p_r + p_r \otimes 1, \quad \varepsilon(p_r) := 0, \quad \text{and} \quad S(p_r) := -p_r \tag{16}$$

completely determine the Hopf structure of $\Lambda[t]$.

Proposition 10. *For $r > 0$,*

$$\begin{aligned} \Delta(e_r) &= \sum_{k=0}^{r} e_k \otimes e_{r-k} & \Delta(s_r) &= \sum_{k=0}^{r} s_k \otimes s_{r-k} & \Delta(q_r) &= \sum_{k=0}^{r} q_k \otimes q_{r-k} \\ S(e_r) &= (-1)^r s_r & S(s_r) &= (-1)^r e_r & S(q_r) &= \sum_{\lambda \vdash r} c_\lambda P_\lambda. \end{aligned}$$

where c_λ is given by Lemma 9.

Proof. Equalities for e_r and s_r are elementary consequences of (16) and may be found in [9, §I.5, Example 25]. The coproduct formula for q_r is (2) in [9, §III.5, Example 8]. The antipode formula for q_r is identical to Lemma 9, as the fundamental morphism ω and the antipode S are related by $S(h) = (-1)^r\omega(h)$ on homogeneous elements h of degree r. □

It happens that $\Lambda[t]$ is self-dual as a Hopf algebra. This may be deduced from Example 8 in [9, §III.5], but we illustrate it here in the power sum basis for the reader not versed in Hopf formalism.

Lemma 11. *The Hopf algebra $\Lambda[t]$ is self-dual with the extended Hall scalar product.*

Proof. Write p^*_λ for $z_\lambda(t)^{-1}p_\lambda$. It is sufficient to check that

$$\big\langle p_\lambda, p^*_\mu \cdot p^*_\nu\big\rangle = \big\langle \Delta(p_\lambda), p^*_\mu \otimes p^*_\nu\big\rangle \quad \text{and} \quad \langle p_\mu \otimes p_\nu, \Delta(p^*_\lambda)\rangle = \langle p_\mu \cdot p_\nu, p^*_\lambda\rangle$$

for all partitions λ, μ, and ν.

Products and coproducts in the power sum basis. Given partitions $\lambda = \langle 1^{m_1}, 2^{m_2}, \cdots\rangle$ and $\mu = \langle 1^{n_1}, 2^{n_2}, \cdots\rangle$, we write $\lambda \cup \mu$ for the partition $\langle 1^{m_1+n_1}, 2^{m_2+n_2}, \cdots\rangle$. Also, we write $\mu \leq \lambda$ if $n_i \leq m_i$ for all $i \geq 1$. In this case, we define

$$\binom{\lambda}{\mu} = \prod_{i\geq 1} \binom{m_i}{n_i},$$

and otherwise define $\binom{\lambda}{\mu} = 0$. Since the power sum basis is multiplicative ($p_\lambda = \prod_{i\geq 1} p_{\lambda_i}$), we have $p_\mu \cdot p_\nu = p_{\mu\cup\nu}$. Since Δ is an algebra map, the first formula in (16) gives

$$\Delta(p_\lambda) = \sum_{\substack{\mu\leq\lambda \\ \mu\cup\nu=\lambda}} \binom{\lambda}{\mu} p_\mu \otimes p_\nu \,.$$

Products and coproducts in dual basis. It is easy to see that

$$z_\lambda(t)^{-1} \cdot \binom{\lambda}{\mu} = z_\mu(t)^{-1} \cdot z_\nu(t)^{-1} \tag{17}$$

whenever $\nu \cup \mu = \lambda$. Using (17) and the formulas for product and coproduct in the power sum basis, we deduce that

$$p^*_\mu \cdot p^*_\nu = \binom{\mu\cup\nu}{\mu} p^*_{\mu\cup\nu}\,, \quad \text{and} \quad \Delta(p^*_\lambda) = \sum_{\substack{\mu\leq\lambda \\ \mu\cup\nu=\lambda}} p^*_\mu \otimes p^*_\nu \,.$$

Checking the desired identities. Using the preceding formulas, we get

$$\big\langle \Delta(p_\lambda), p^*_\mu \otimes p^*_\nu\big\rangle \;=\; \binom{\lambda}{\mu} \cdot \delta_{\lambda,\mu\cup\nu} \;=\; \big\langle p_\lambda, p^*_\mu \cdot p^*_\nu\big\rangle \,.$$

and

$$\langle p_\mu \cdot p_\nu, p^*_\lambda\rangle \;=\; \delta_{\lambda,\mu\cup\nu} \;=\; \langle p_\mu \otimes p_\nu, \Delta(p^*_\lambda)\rangle \,.$$

This completes the proof of the lemma. □

After (13), (14) and Lemma 11, we see that $P_{\lambda/\mu} = Q_\mu \rightharpoonup P_\lambda$ and $Q_{\lambda/\mu} = P_\mu \rightharpoonup Q_\lambda$.

3. Proofs of the main theorems

We specialize (15) to Hall–Littlewood polynomials, putting $a \rightharpoonup g = P_{\lambda/\mu}$.

Proof of Theorem 2. Taking $h = e_r$ in (15), we get

$$P_{\lambda/\mu} \cdot e_r = (Q_\mu \rightharpoonup P_\lambda) \cdot e_r = \sum_{(e_r)} \big(S(e_r'') \rightharpoonup Q_\mu\big) \rightharpoonup \big(P_\lambda \cdot e_r'\big) \tag{18}$$

$$= \sum_{k=0}^{r} (S(e_k) \rightharpoonup Q_\mu) \rightharpoonup \big(P_\lambda \cdot e_{r-k}\big) \tag{19}$$

$$= \sum_{k=0}^{r} (-1)^k \,(s_k \rightharpoonup Q_\mu) \rightharpoonup \big(P_\lambda \cdot e_{r-k}\big) \tag{20}$$

$$= \sum_{k=0}^{r} (-1)^k \left(\sum_{\tau} t^{n(\tau)} Q_{\mu/\tau}\right) \rightharpoonup \big(P_\lambda \cdot e_{r-k}\big) \tag{21}$$

$$= \sum_{k=0}^{r} (-1)^k \left(\sum_{|\mu/\mu^-|=k} \left(\sum_{\tau} t^{n(\tau)} f^{\,\mu}_{\mu^-,\tau}(t)\right) Q_{\mu^-}\right) \rightharpoonup \left(\sum_{|\lambda^+/\lambda|=r-k} \mathrm{vs}_{\lambda^+/\lambda}(t) P_{\lambda^+}\right) \tag{22}$$

$$= \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|} \,\mathrm{sk}_{\mu/\mu^-}(t)\, \mathrm{vs}_{\lambda^+/\lambda}(t) P_{\lambda^+/\mu^-}\,. \tag{23}$$

For (19) and (20), we used Proposition 10. For (21), we expanded s_k in the P basis (cf. the proof of Corollary 6) and used the Hopf characterization of skew elements. Explicitly,

$$s_k \rightharpoonup Q_\mu = \left(\sum_{\tau \vdash k} t^{n(\tau)} P_\tau\right) \rightharpoonup Q_\mu = \sum_{\tau \vdash k} t^{n(\tau)} Q_{\mu/\tau}\,.$$

We use (3) and (13) to pass from (21) to (22): the coefficient of Q_{μ^-} in the expansion of $Q_{\mu/\tau}$ is equal to the coefficient of P_μ in $P_{\mu^-} P_\tau$. Finally, (23) follows from Corollary 6. □

Proof of Theorem 3. Taking $h = s_r$ in (15), we get

$$P_{\lambda/\mu} \cdot s_r = (Q_\mu \rightharpoonup P_\lambda) \cdot s_r = \sum_{(s_r)} \big(S(s_r'') \rightharpoonup Q_\mu\big) \rightharpoonup \big(P_\lambda \cdot s_r'\big) \tag{24}$$

$$= \sum_{k=0}^{r} (S(s_k) \rightharpoonup Q_\mu) \rightharpoonup \big(P_\lambda \cdot s_{r-k}\big) \tag{25}$$

$$= \sum_{k=0}^{r} (-1)^k \,(e_k \rightharpoonup Q_\mu) \rightharpoonup \big(P_\lambda \cdot s_{r-k}\big) \tag{26}$$

$$= \sum_{k=0}^{r} (-1)^k Q_{\mu/1^k} \rightharpoonup \big(P_\lambda \cdot s_{r-k}\big) \tag{27}$$

$$= \sum_{k=0}^{r} (-1)^k \left(\sum_{|\mu/\mu^-|=k} \mathrm{vs}_{\mu/\mu^-}(t) Q_{\mu^-}\right) \rightharpoonup \left(\sum_{|\lambda^+/\lambda|=r-k} \mathrm{sk}_{\lambda^+/\lambda}(t) P_{\lambda^+}\right) \tag{28}$$

$$= \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|} \,\mathrm{vs}_{\mu/\mu^-}(t)\, \mathrm{sk}_{\lambda^+/\lambda}(t) P_{\lambda^+/\mu^-}\,. \tag{29}$$

For (25) and (26), the proof is the same as above. For (27), we used $e_k = P_{1^k}$, while for (28), we used (3) and (5). Equation (29) is obvious. □

Proof of Theorem 4. We present two proofs. The first is along the lines of the preceding proofs of Theorems 2 and 3. Taking $h = s_r$ in (15), we get

$$
\begin{aligned}
P_{\lambda/\mu} \cdot q_r = (Q_\mu \rightharpoonup P_\lambda) \cdot q_r &= \sum_{(q_r)} \big(S(q_r{}'') \rightharpoonup Q_\mu\big) \rightharpoonup \big(P_\lambda \cdot q_r{}'\big) && (30)\\
&= \sum_{k=0}^{r} \left(S(q_k) \rightharpoonup Q_\mu\right) \rightharpoonup \big(P_\lambda \cdot q_{r-k}\big) && (31)\\
&= \sum_{k=0}^{r} \left(\sum_{\tau\vdash k} c_\tau(t) P_\tau \rightharpoonup Q_\mu\right) \rightharpoonup \big(P_\lambda \cdot q_{r-k}\big) && (32)\\
&= \sum_{k=0}^{r} \left(\sum_{\tau\vdash k} c_\tau(t) Q_{\mu/\tau}\right) \rightharpoonup \big(P_\lambda \cdot q_{r-k}\big) && (33)\\
&= \sum_{k=0}^{r} \left(\sum_{|\mu/\mu^-|=k} \left(\sum_{\tau} c_\tau(t) f^{\,\mu}_{\mu^-,\tau}(t)\right) Q_{\mu^-}\right) \rightharpoonup \left(\sum_{|\lambda^+/\lambda|=r-k} \mathrm{hs}_{\lambda^+/\lambda}(t) P_{\lambda^+}\right) && (34)\\
&= \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|}(-t)^{|\tau/\mu^-|}\, \mathrm{vs}_{\mu/\tau}(t)\, \mathrm{sk}_{\tau/\mu^-}\, \mathrm{hs}_{\lambda^+/\lambda}(t) P_{\lambda^+/\mu^-}\,. && (35)
\end{aligned}
$$

The only line that needs a comment is (35).

Substitute $y = -1/t$, $\lambda = \mu$, $\mu = \mu^-$ and $\nu = \tau$ into Theorem 7. We get

$$\sum_{\tau} \mathrm{vs}_{\mu/\tau}(t)\, \mathrm{sk}_{\tau/\mu^-}(t)(-1/t)^{|\mu/\tau|} = \sum_{\sigma} t^{n(\sigma)-\binom{\ell(\sigma)}{2}} f^{\mu}_{\tau,\mu^-}(t) \prod_{j=1}^{\ell(\sigma)} (-1/t + t^{j-1}),$$

and, after multiplying by $t^{|\mu/\mu^-|}$,

$$\sum_{\tau} (-1)^{|\mu/\tau|} t^{|\tau/\mu^-|}\, \mathrm{vs}_{\mu/\tau}(t)\, \mathrm{sk}_{\tau/\mu^-}(t) = \sum_{\sigma} t^{n(\sigma)-\binom{\ell(\sigma)}{2}+|\mu/\mu^-|-\ell(\sigma)} f^{\mu}_{\tau,\mu^-}(t) \prod_{j=1}^{\ell(\sigma)} (-1 + t^{j}).$$

Now $|\mu/\mu^-| = |\sigma|$ and $n(\sigma) - \binom{\ell(\sigma)}{2} + |\sigma| - \ell(\sigma) = \sum_i \left(\binom{\sigma_i'}{2} + \sigma_i'\right) - \binom{\sigma_1'+1}{2} = \sum_{i=2}^{\sigma_1} \binom{\sigma_i'+1}{2}$, which shows that

$$\sum_{\sigma} c_\sigma f^{\mu}_{\sigma,\mu^-}(t) = \sum_{\tau} (-1)^{|\mu/\tau|} t^{|\tau/\mu^-|}\, \mathrm{vs}_{\mu/\tau}(t)\, \mathrm{sk}_{\tau/\mu^-}(t),$$

with the sum over all τ satisfying $\mu^- \subseteq \tau \subseteq \mu$. This completes the first proof.

The second proof uses Theorems 1, 2 and 3. Recall from (10) that $q_r = \sum_{k=0}^{r} (-t)^k s_{r-k} e_k$. We have

$$
\begin{aligned}
P_{\lambda/\mu} \cdot q_r = P_{\lambda/\mu} \cdot \left(\sum_{k=0}^{r} (-t)^k s_{r-k} e_k\right) &= \sum_{k=0}^{r} (-t)^k (P_{\lambda/\mu} s_{r-k}) e_k\\
&= \sum_{k=0}^{r} (-t)^k \sum_{\sigma,\tau} (-1)^{|\mu/\tau|}\, \mathrm{vs}_{\mu/\tau}(t)\, \mathrm{sk}_{\sigma/\lambda}(t) P_{\sigma/\tau} e_k
\end{aligned}
$$

$$= \sum_{\sigma,\tau,\mu^-,\lambda^+} (-t)^{|\tau/\mu^-|+|\lambda^+/\sigma|}(-1)^{|\mu/\tau|+|\tau/\mu^-|} \operatorname{vs}_{\mu/\tau}(t) \operatorname{sk}_{\sigma/\lambda}(t) \operatorname{sk}_{\tau/\mu^-}(t) \operatorname{vs}_{\lambda^+/\sigma}(t) P_{\lambda^+/\mu^-}$$

$$= \sum_{\tau,\mu^-,\lambda^+} (-1)^{|\mu/\mu^-|}(-t)^{|\tau/\mu^-|} \operatorname{vs}_{\mu/\tau}(t) \operatorname{sk}_{\tau/\mu^-}(t) \left(\sum_{\sigma} (-t)^{|\lambda^+/\sigma|} \operatorname{vs}_{\lambda^+/\sigma}(t) \operatorname{sk}_{\sigma/\lambda}(t) \right) P_{\lambda^+/\mu^-}$$

$$= \sum_{\tau,\mu^-,\lambda^+} (-1)^{|\mu/\mu^-|}(-t)^{|\tau/\mu^-|} \operatorname{vs}_{\mu/\tau}(t) \operatorname{sk}_{\tau/\mu^-}(t) \operatorname{hs}_{\lambda^+/\lambda}(t) P_{\lambda^+/\mu^-},$$

where we used Lemma 5 in the final step. □

Our final result is on the uniqueness of the expansions.

Theorem 12. *Let $a_{\lambda/\mu}(t)$ and $b_{\lambda/\mu}(t)$ be polynomials defined for $\lambda \supseteq \mu$, with $b_{\emptyset/\emptyset}(t) = 1$. For fixed $\lambda \supseteq \mu$ and $r \geq 0$, consider the expression*

$$\mathcal{E}_{\lambda,\mu,r} = \sum_{\substack{\lambda^+ \supseteq \lambda,\, \mu^- \subseteq \mu \\ |\lambda^+/\lambda| + |\mu/\mu^-| = r}} (-1)^{|\mu/\mu^-|} a_{\lambda^+/\lambda}(t) b_{\mu/\mu^-}(t) P_{\lambda^+/\mu^-}.$$

1) If $\mathcal{E}_{\lambda,\mu,r} = P_{\lambda/\mu}\, s_{1^r}$ $\forall \lambda,\mu,r$ then $a_{\lambda^+/\lambda} = \operatorname{vs}_{\lambda^+/\lambda}$ and $b_{\mu/\mu^-} = \operatorname{sk}_{\mu/\mu^-}$.
2) If $\mathcal{E}_{\lambda,\mu,r} = P_{\lambda/\mu}\, s_r$ $\forall \lambda,\mu,r$ then $a_{\lambda^+/\lambda} = \operatorname{sk}_{\lambda^+/\lambda}$ and $b_{\mu/\mu^-} = \operatorname{vs}_{\mu/\mu^-}$.
3) If $\mathcal{E}_{\lambda,\mu,r} = P_{\lambda/\mu}\, q_r$ $\forall \lambda,\mu,r$ then $a_{\lambda^+/\lambda} = \operatorname{hs}_{\lambda^+/\lambda}$ and $b_{\mu/\mu^-} = \sum_\nu (-t)^{|\nu/\mu^-|} \operatorname{vs}_{\mu/\nu} \operatorname{sk}_{\nu/\mu^-}$.

Proof. We prove only the first statement, the others being similar. Suppose that we have

$$P_{\lambda/\mu}\, s_{1^r} = \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|} a_{\lambda^+/\lambda}(t) b_{\mu/\mu^-}(t) P_{\lambda^+/\mu^-}.$$

If we set $\mu = \emptyset$, we get the expansion of $P_\lambda s_{1^r}$ over (non-skew) Hall-Littlewood polynomials, which is, of course, unique. Therefore $a_{\lambda/\mu}(t)\, b_{\emptyset/\emptyset}(t) = a_{\lambda/\mu}(t) = \operatorname{vs}_{\lambda/\mu}(t)$ for all $\lambda \supseteq \mu$. We will prove by induction on $|\lambda/\mu|$ that $b_{\lambda/\mu}(t) = \operatorname{sk}_{\lambda/\mu}(t)$. For $\lambda = \mu$ and $r = 0$, we get $P_{\lambda/\lambda} = b_{\lambda/\lambda}(t) P_{\lambda/\lambda}$, so $b_{\lambda/\lambda}(t) = 1 = \operatorname{sk}_{\lambda/\lambda}(t)$. Suppose that $b_{\lambda/\mu}(t) = \operatorname{sk}_{\lambda/\mu}(t)$ for $|\lambda/\mu| < r$ and that $|\lambda/\mu| = r$. Take

$$\sigma = (\underbrace{\lambda_1 + \mu_1, \ldots, \lambda_1 + \mu_1}_{\ell(\lambda)}, \lambda_1 + \mu_1, \lambda_1 + \mu_2, \ldots, \lambda_1 + \mu_{\ell(\mu)})$$

$$\tau = (\underbrace{\lambda_1 + \mu_1, \ldots, \lambda_1 + \mu_1}_{\ell(\lambda)}, \underbrace{\lambda_1, \ldots, \lambda_1}_{\ell(\mu)}).$$

Note that $\lambda \subseteq \sigma$. Also, the diagram of σ/τ is a translation of the diagram of μ. That means there is only one LR-sequence S (see [9, p. 185]) of shape σ/τ, and it has type μ. This implies that $f^\sigma_{\tau,\mu} = f_S(t)$, $f^\sigma_{\tau,\mu'} = 0$ for $\mu \neq \mu'$ (see [9, pp. 194 and 218]). Therefore $P_{\sigma/\tau}$ is a non-zero polynomial multiple of P_μ. Now

$$\begin{aligned} P_{\sigma/\lambda}\, s_{1^r} &= \sum_{\sigma^+,\lambda^-} (-1)^{|\lambda/\lambda^-|} a_{\sigma^+/\sigma}(t) b_{\lambda/\lambda^-}(t) P_{\sigma^+/\lambda^-} \\ &= \sum_{\sigma^+,\lambda^-} (-1)^{|\lambda/\lambda^-|} \operatorname{vs}_{\sigma^+/\sigma}(t) b_{\lambda/\lambda^-}(t) P_{\sigma^+/\lambda^-} \\ &= \sum_{\sigma^+,\lambda^-} (-1)^{|\lambda/\lambda^-|} \operatorname{vs}_{\sigma^+/\sigma}(t) \operatorname{sk}_{\lambda/\lambda^-}(t) P_{\sigma^+/\lambda^-}, \end{aligned}$$

where we used Theorem 2. By the induction hypothesis, $b_{\lambda/\lambda^-}(t) = \mathrm{sk}_{\lambda/\lambda^-}(t)$ if $|\lambda/\lambda^-| < r$. After cancellations, we get

$$\sum_{\lambda^-}(-1)^{|\lambda/\lambda^-|}(b_{\lambda/\lambda^-}(t) - \mathrm{sk}_{\lambda/\lambda^-}(t))P_{\sigma/\lambda^-} = 0,$$

where the sum on the left is over all $\lambda^- \subseteq \lambda$ such that $|\lambda/\lambda^-| = r$. Now take scalar product with Q_τ. Since $\langle P_{\sigma/\lambda^-}, Q_\tau\rangle = \langle P_\sigma, Q_{\lambda^-}Q_\tau\rangle = \langle P_{\sigma/\tau}, Q_{\lambda^-}\rangle$ is the coefficient of P_{λ^-} in $P_{\sigma/\tau}$, we see that $(-1)^{|\lambda/\mu|}(b_{\lambda/\mu}(t) - \mathrm{sk}_{\lambda/\mu}(t)) = 0$. That is, $b_{\lambda/\mu}(t) = \mathrm{sk}_{\lambda/\mu}(t)$. $\square$

Remark. Similar proofs show that the expansions of $s_{\lambda/\mu}s_{1^r}$, $s_{\lambda/\mu}s_r$ and $s_{\lambda/\mu}P_r$ in terms of skew Schur functions are also unique in the sense of Theorem 12, a fact that was not noted in either [3] or [6].

Remark. It would be preferable to have a simpler expression for the polynomial

$$b_{\lambda/\mu}(t) = \sum_{\nu}(-t)^{|\nu/\mu|}\,\mathrm{vs}_{\lambda/\nu}(t)\,\mathrm{sk}_{\nu/\mu}(t) \tag{36}$$

from Theorems 4 and 12(3), i.e., one involving only the boxes of λ/μ in the spirit of $\mathrm{hs}_{\lambda/\mu}(t)$, so that we could write

$$P_{\lambda/\mu}\cdot q_r = \sum_{\lambda^+,\mu^-}(-1)^{|\mu/\mu^-|}\,\mathrm{hs}_{\lambda^+/\lambda}(t)b_{\mu/\mu^-}(t)\,P_{\lambda^+/\mu^-}\,,$$

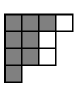
where the sum is over all $\lambda^+ \supseteq \lambda$, $\mu^- \subseteq \mu$ such that $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

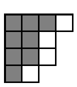
Toward this goal, we point out a hidden symmetry in the polynomials $b_{\lambda/\mu}(\mathrm{t})$. Writing q_r as $\sum_{k=0}^r(-t)^k e_k s_{r-k}$ before running through the second proof of Theorem 4 (i.e., applying Theorems 2 and 3 in the reverse order) reveals

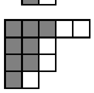
$$b_{\lambda/\mu}(t) = \sum_{\nu}(-t)^{|\lambda/\nu|}\,\mathrm{sk}_{\lambda/\nu}(t)\ \mathrm{vs}_{\nu/\mu}(t)\,. \tag{37}$$

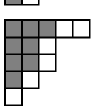
Further toward this goal, note how similar (36) is to the sum in Lemma 5, which reduces to the tidy product of polynomials $\mathrm{hs}_{\lambda/\mu}(t)$.

Basic computations suggest some hint of a polynomial-product description for $b_{\lambda/\mu}(t)$,

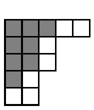
 $:\ \ -(t-1)^2(t+1)\left(t^3+t^2+t-1\right)$

 $:\ \ (t-1)^2(t+1)\left(t^3+t^2+t-1\right)^2$

 $:\ \ t(t-1)^2(t+1)\left(t^3+t^2+t-1\right)^2$

 $:\ \ t(t-1)^2(t+1)\left(t^2+t-1\right)\left(t^3+t^2+t-1\right)^2,$

but others suggest that such a description will not be tidy,

 $:\ \ -t^2(t-1)^2(t+1)^2\left(t^3+t^2+t-1\right)\left(t^7+t^6+2t^5-t^3-2t^2-t+1\right).$

We leave a concise description of the $b_{\lambda/\mu}(t)$ as an open problem.

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